In conventional stellarator neoclassical theory, the transport coefficients are determined so as to fulfill the so-called ambipolarity constraint on the (flux-surface-averaged) radial particle fluxes but without regard to the additional requirement that the underlying solutions of the kinetic equation also satisfy (local) quasi-neutrality. This is done principally for the sake of simplicity as it allows one to assume that density, \( n \), and electrostatic potential, \( \Phi \), are constant on a flux surface. As justification, the expected variation of \( \Phi \) on the flux surface has been determined for a handful of analytically tractable cases and been shown to be of the same magnitude as the variation of the magnetic-field strength, \( B \). Both variations lead to radial particle motion through the \( \nabla \Phi \times B \) and \( \nabla B \times B \) drifts, respectively, but the latter weighs more heavily in the determination of neoclassical transport coefficients due to its stronger energy dependence. It was thus concluded that the expected variation of \( \Phi \) should have only a modest impact on bulk-plasma transport [1-3]. It has been subsequently pointed out, however, that the nonlinear nature of the ambipolarity and quasi-neutrality constraints may allow for multiple solutions, some of which predict quite different plasma confinement from those of ambipolarity alone [4].

In this contribution, the implications which the quasi-neutrality condition has for neoclassical transport in stellarators are investigated using a version of the General Solution of the Ripple-Averaged Kinetic Equation (GSRAKE) [5], upgraded to account for the variation of \( \Phi \) on flux surfaces (including compressional effects on the poloidal rotation). Use of such a numerical tool obviates the necessity of any asymptotic ordering of the kinetic equation and allows the consideration of more realistic magnetic fields, including those with a high degree of \( (\nabla B) \) drift optimization. Radial profiles of quantities of interest are determined for the Large Helical Device (LHD) and Wendelstein 7-X (W7-X), assuming specified density and temperature profiles for hydrogen plasmas.

Basic Equations and Assumptions

The steady-state drift-kinetic equation may be written \[ \sum_i \left( \frac{dz^i}{dt} \right)(\partial f/\partial z^i) = C(f) \] in which \( f \) is the distribution function, \( C \) the collision operator and the \( z^i \) represent the coordinates chosen to describe the 5D phase space. For stellarators it is advantageous to choose toroidal flux coordinates as the spatial variables: \( r \) the flux-surface label, \( \theta \) the poloidal angle and \( \zeta = N\phi \) where \( N \) is the field period number and \( \phi \) the toroidal angle. A number of choices are available for the velocity-space variables; initially the kinetic energy, \( \kappa = \frac{mv^2}{2} \), and the magnetic moment, \( \mu = \frac{mv^2}{2B} \), are chosen here. Within neoclassical theory, it is common practice to seek solutions of the kinetic equation under the assumption that the transport is radially local. In keeping with this ansatz, the distribution function is expanded \( f = f_0 + f_1 \), where the largest portion, \( f_0 \), is the solution of the kinetic equation for \( dr/dt = 0 \) and the much smaller \( f_1 \) contains all effects due to particles’ radial drifts. Employing a similar logic, the density and electrostatic potential are expanded \( n = n_0(r) + n_1(r, \theta, \zeta) \)
and $\Phi = \Phi_0(r) + \Phi_1(r, \theta, \zeta)$, where the first-order terms are again attributable to the particles’ radial motion and thus assumed to be small. For the temperature, however, the very large parallel thermal conductivities of fusion plasmas are invoked to justify taking $T = T(r)$. Explicit forms of the drift equations are not necessary at this point; it is sufficient to note that the total energy, $E = \kappa + q\Phi$, is a constant ($q$ is the particle charge) to obtain an expression for $d\kappa/dt$. The lowest-order solution for the distribution function is then easily shown to be

$$f_0 = n_0 \left( \frac{m}{2\pi T} \right)^{3/2} \exp\left(-\left(\kappa + q\Phi_1\right)/T\right) = f_m \exp(-q\Phi_1/T)$$

i.e. the usual Maxwellian multiplied by the Boltzmann factor. In formulating the kinetic equation for $f_1$ it is useful to replace the kinetic energy with $\kappa + q\Phi_1$. In terms of the new variable, the first-order kinetic equation is expressed

$$C_\mu(f_1) - \frac{d\theta}{dt} \frac{\partial f_1}{\partial \theta} - \frac{d\zeta}{dt} \frac{\partial f_1}{\partial \zeta} = \frac{dr}{dt} \left\{ \frac{1}{n_0} \frac{\partial n_0}{\partial r} + \frac{q}{T} \frac{\partial \Phi_0}{\partial r} + \left(\frac{\kappa_0}{T} - \frac{3}{2}\right) \frac{1}{T} \frac{\partial T}{\partial r} \right\} f_0 = \frac{dr}{dt} \frac{\partial f_0}{\partial r} \bigg|_\varepsilon$$

in which both $r$ and $\kappa_0$ have now become mere parameters. To eliminate a further variable from the problem, one makes use of the (nearly) periodic behavior of collisionless particle orbits relative to the local “helical” ripples of the stellarator’s magnetic field. A clean time-scale separation is facilitated by replacing the variables $(\theta, \mu)$ with $(\theta_n, k^2)$, the local field-line and pitch-angle variables, respectively; see reference [6] for more detail. Application of the ripple average (more commonly known as the bounce average when only localized particles are considered) to the drift-kinetic equation results in

$$\left\langle C_{k^2}(f_1) \right\rangle - \left\langle \frac{d\theta_n}{dt} \right\rangle \frac{\partial f_1}{\partial \theta_n} - \left\langle \frac{dk^2}{dt} \right\rangle \frac{\partial f_1}{\partial k^2} = \left\langle \frac{dr}{dt} \right\rangle \frac{\partial f_0}{\partial r} \bigg|_\varepsilon$$

where $\langle A \rangle$ denotes the ripple average of the quantity $A$. For the radial drift velocity, one has

$$\left\langle \frac{dr}{dt} \right\rangle = \frac{\partial \theta_n}{\partial \theta} \left\{ \frac{1}{rB_0} \frac{\partial \Phi_1}{\partial \theta_n} + \frac{v_d}{\epsilon_i} \left(1 + p^2\right) \frac{1}{B} \frac{\partial B}{\partial \theta_n} \right\}$$

with $v_d = \kappa/(qR_0B_0)$, $\epsilon_i = r/R_0$, $p = v_\parallel/v$ and where $R_0$ is the major radius. The ripple averages of the remaining quantities are given in reference [6].

To solve this *ripple-averaged* kinetic equation, GSRAKE expands the $\theta_n$ dependence of $f_1$ in a Fourier series and calculates the $k^2$ dependence of the Fourier coefficients using finite-difference techniques; the boundary conditions are formulated to insure continuity of the phase-space fluxes at the interface between regions in which localized and non-localized particles are found [5]. The neoclassical particle flux is determined from

$$\Gamma_\alpha = -D^\alpha_n n_0 \left\{ \frac{1}{n_0} \frac{\partial n_0}{\partial r} - \frac{q^\alpha E_r}{T^\alpha} + \left(\frac{D^\alpha_2}{D^\alpha_1} - \frac{3}{2}\right) \frac{1}{T^\alpha} \frac{\partial T^\alpha}{\partial r} \right\}$$

where $\alpha$ is the species index ($e$ for electron, $i$ for ion), $E_r = -\partial \Phi_0/\partial r$ is the radial electric field and the transport coefficients are given by

$$D^\alpha_n \equiv \frac{2}{\sqrt{\pi}} \int_0^\infty dx \ x^{n-1/2} D(x) e^{-x}$$
with \( x = \kappa_0 / T^\alpha \) the normalized energy and where
\[
D = \frac{1}{2\pi} \mathcal{F} \int_0^{2\pi} d\theta_n \int_0^\infty dk^2 J \left( \frac{dr}{dt} \right) f_1
\]
is the mono-energetic diffusion coefficient. In this expression, \( \mathcal{F} \) normalizes the flux-surface average in magnetic coordinates and \( J \) is the Jacobian for the ripple-averaged phase space.

The density variation on the flux surface is determined from
\[
n_i^\alpha = n_0^\alpha \exp \left( -q^\alpha \Phi_1 / T^\alpha \right) - n_0^\alpha + \int d^3v \, f_1^\alpha
\]
and then Fourier decomposed according to
\[
n_i^\alpha = \sum_{l=1}^L (n_i^\alpha)^\ell \cos \ell \theta_n + \sum_{l=1}^L (n_i^\alpha)^\ell \sin \ell \theta_n.
\]
Quasi-neutrality requires the equality of each of the individual Fourier coefficients for ions and electrons through \( \ell = L \). This is achieved by iterative adjustments of the Fourier coefficients of the electrostatic potential
\[
\Phi_1 = \sum_{l=1}^L \Phi_1^\ell \cos \ell \theta_n + \sum_{l=1}^L \Phi_1^\ell \sin \ell \theta_n
\]
along with the radial electric field to simultaneously satisfy the ambipolarity constraint \( \Gamma^e = \Gamma^i \), using standard methods for solving systems of non-linear equations.

**Results**

As a first example, the standard configuration of the LHD \((R_0 = 3.75 \, m)\) has been considered, assuming \( n_0^e = n_0^i = 3.2 \times 10^{18} (1 - \rho^8) \, m^{-3} \) and \( T^e = T^i = 2(1 - \rho^2) \, keV \) [7] where \( \rho = r/a \) is the normalized flux-surface radius. To perform neoclassical transport analysis, the usual practice is to ignore quasi-neutrality (assume \( \Phi_1 = 0 \)) and to use a simple root-finding method to determine the value(s) of \( E_r \) at which the ambipolarity constraint is satisfied. Results obtained in this manner are indicated in figure 1 by red symbols; the circles mark ion-root solutions, the squares are used for electron roots and triangles indicate the thermodynamically unstable solution. Profiles of \( E_r \) are plotted in the first row on the left, the ambipolar particle fluxes are shown to their right. The neoclassical energy fluxes
\[
Q^\alpha = -\frac{n_0^\alpha B_0 T^\alpha}{4\pi^2 r \mathcal{F}} \int_{-\pi}^{\pi} d\zeta \int_0^{2\pi} d\theta_n \frac{1}{B^2} \frac{\partial \Phi_1}{\partial \theta_n} \exp \left( -q^\alpha \Phi_1 / T^\alpha \right) [\frac{1}{n_0^\alpha} \frac{\partial n_0^\alpha}{\partial r} + \frac{q^\alpha E_r}{T^\alpha} + \left( \frac{D_3^\alpha}{D_2^\alpha} - \frac{3}{2} \right) \frac{1}{T^\alpha} \frac{\partial T^\alpha}{\partial r}]
\]
are given in the second row for electrons (left) and ions (right). In the third row, the radial profiles of the first two Fourier components of \( n_i^\alpha / n_0^\alpha \) are plotted; this is repeated for ions in the fourth row.

The calculations are then carried out a second time, requiring that the ambipolarity and quasi-neutrality constraints both be fulfilled. The results are depicted in figure 1 by the black
Figure 1. Results for the standard LHD configuration as described in the text.
symbols; in the fifth row the radial profiles of the first two Fourier components of $\Phi_1$ are given. A comparison of the two sets of results shows a significant reduction in the radial extent of the region where multiple solutions for $E_r$ exist when quasi-neutrality is required. Additionally, both particle and energy fluxes are increased considerably (by as much as a factor of two) at outer radii where the density and temperature gradients are large. Most remarkable, however, are the results for $n_1$, the cosine component alone yielding a density variation on the flux surface of as much as 18%.

As a second example, the high-mirror configuration of W7-X is considered. Here the density and temperature profiles are taken from 1D transport simulations of discharges with strong central ECRH [8] and are depicted in figure 2. The results obtained are plotted in figure 3. W7-X is strongly drift optimized and as a consequence the variation of the density on the flux surface has a maximum value of only 2% for the chosen simulation parameters. Particle and energy fluxes are increased by 20% at most in the gradient region and are actually decreased near the plasma center where the electron temperature profile is responsible for a strong electron root.

In both LHD and W7-X simulations, extensive searches were made for additional sets of $(E_r, \Phi_1)$ values which also satisfy the ambipolarity and quasi-neutrality conditions. Success was encountered only in a very small number of cases in which the higher-order Fourier modes of $\Phi_1$ appear with larger magnitudes, but the resulting effect on the transport is small.

![Figure 2. Density and temperature profiles used for the high-mirror W7-X calculations are given. For $n_0$ the profiles of electrons and ions are identical; $T^e$ is plotted in red, $T^i$ in blue.](image-url)

Figure 3. Results for the high-mirror W7-X configuration are plotted as in figure 1.