Plasma Phys. Control. Fusion **47** (2005) B743–B754 doi:10.1088/0741-3335/47/12B/S56

The foundations of diffusion revisited

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Received 1 July 2005
Published 11 November 2005
Online at stacks.iop.org/PPCF/47/B743

Abstract
Diffusion is essentially the macroscopic manifestation of random (Brownian) microscopic motion. This idea has been generalized in the continuous time random walk formalism, which under quite general conditions leads to a generalized master equation (GME) that provides a useful modelling framework for transport. Here we review some of the basic ideas underlying this formalism from the perspective of transport in (magnetic confinement) plasmas.

Under some specific conditions, the fluid limit of the GME corresponds to the Fokker–Planck (FP) diffusion equation in inhomogeneous systems, which reduces to Fick's law when the system is homogeneous. It is suggested that the FP equation may be preferable in fusion plasmas due to the inhomogeneity of the system, which would imply that part of the observed inward convection ('pinch') can be ascribed to this inhomogeneity.

The GME also permits a mathematically sound approach to more complex transport issues, such as the incorporation of critical gradients and non-local transport mechanisms. A toy model incorporating these ingredients was shown to possess behaviour that bears a striking similarity to certain unusual phenomena observed in fusion plasmas.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

On the 100th anniversary of the publication of the famous paper by Einstein [1] in which the relation between Brownian motion and macroscopic diffusion was established, it is appropriate to review the basic concepts of diffusion in the light of recent developments. The idea that diffusion is the macroscopic manifestation of predominantly random microscopic motion (Brownian motion) was generalized in the mid-twentieth century in what is known as the continuous time random walk (CTRW) formalism [2]. Within this framework, a much wider range of types of random motion is allowed than only the standard (Gaussian or Brownian) random walk [3]. In fact, the starting point is the statement that a particle, located at the space-time point (x', t'), has a certain probability to move to a new point (x, t), with $t > t'$, given by the probability density function (pdf):

$$\xi(x - x', x'; t - t', t'). \tag{1}$$

This pdf is normalized to guarantee conservation of probability:

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_{0}^{\infty} \mathrm{d}\tau \xi(x - x', x'; \tau, t') = 1. \tag{2}$$

As a side remark, we note that in this context the word 'particle' is used in a generic sense. Depending on the problem at hand, it may refer to real particles (e.g. electrons or impurities) or to any spatially localized entity liable to be transported, such as fluid mass or energy packets.

The CTRW formalism does not enter into the details of the physical processes that cause the particle motion. It merely ascribes a probability to the *result* of the physical interactions taking place. In simple cases, when a detailed physical model is available for the microscopic motion (the kinetics), these probabilities may be estimated. However, even when such detailed knowledge is unavailable or too complex to even yield approximate solutions, the CTRW framework can be of use. In such cases, CTRW models can be constructed using probability distributions that are consistent with the symmetries of the system, with additional input from incomplete experimental and theoretical knowledge. If the latter is feasible, the resulting CTRW model will provide an approximate statistical description of the transport process, in spite of the lack of detailed knowledge of the microscopic reality, on occasion resulting in extremely useful and practical models that help clarify global transport and scaling behaviour as well as specific complex phenomena. CTRW models are generalizations of the phenomenological laws associated to the names of Fick and Fourier, while allowing the modelling of transport in systems with a complexity far beyond the usual limits, on a mathematically sound basis and complying with all basic conservation requirements (of particles, energy, etc).

In the context of magnetically confined plasmas, the combination of neo-classical and turbulent transport falls precisely in this category of complexity. Therefore, in our opinion, the CTRW framework may provide a novel and useful approach to the transport problem in such plasmas. Initial steps in this direction have already been taken and are reviewed in the current paper. However, the scope of the CTRW modelling framework is of an extraordinary amplitude. In the current paper, we cannot hope to present a comprehensive review and merely present some specific results, corresponding to different levels of approximation, which we feel could be relevant to the study of transport in fusion plasma physics, without any pretence of completeness. The work presented here is unfinished, and at present the ultimate goal of a fully fledged probabilistic transport model is still far away, although many necessary elements seem to be available now.

2. Basics

2.1. Assumptions and starting points

In the present work we will assume that ξ is space-time separable, and we will limit ourselves to Markovian particle motion. The latter implies that no memory of past events is retained in the individual motion, so that the distribution of waiting times between successive particle 'jumps' is an exponential law, with characteristic time $\tau_{\mathrm{D}}(x')$, which may depend on x'. Thus, ξ can be written as

$$\xi(x - x', x'; t - t', t') = \frac{p(x - x'; x', t)}{\tau_{\mathrm{D}}(x')} \exp\left[-\frac{(t - t')}{\tau_{\mathrm{D}}(x')}\right]. \tag{3}$$

We will also assume that no (external) forces are operating on the individual particles, so that $p(\Delta x; x, t) = p(-\Delta x; x, t)$. These choices are quite restrictive, and we note that the CTRW framework allows for other possibilities with interesting consequences. By way of example, a non-Markovian choice for the waiting time distribution leads to subdiffusion [4–6], which can be associated to the physical process of particles sticking to topologically singular sites (e.g. localized eddies). However, in the following it will become clear that these particular choices are rather appropriate for the type of transport dynamics we are concerned with.

The symmetric particle step distribution p can take on a variety of forms, always subject to the normalization requirement. We assume that p describes motion on *mesoscopic* spatial and temporal scales, i.e. on a range of scales between those pertaining to the microscopic movement and the maximum scale set by the system size and the life of the system. This is an important assumption, since it allows one to decouple the model somewhat from the microscopic motion. The model will therefore 'average out' the details of the microscopic kinetics and retain only effects larger than this mesoscale. Thus, we assume that the mesoscopic particle 'jumps' described by the distribution p are in fact the result of the sum of many microscopic random movements. In this case, the central limit theorem predicts that p must be a limiting distribution of the Lévy class [7].

The Gaussian distribution is only a specific element in the Lévy class of distributions and arises from the additional requirement that the width (or standard deviation) of p be a finite number. For many physical situations this is a reasonable requirement, but in recent years it has become clear that in many branches of science non-Gaussian Lévy distributions do indeed occur in practice (cf [8] and references therein). We therefore specifically wish to leave open the possibility that on a mesoscale level transport may involve non-Gaussian Lévy distributions, without however claiming that this necessarily must be so. In the following, we will discuss the consequences of this possibility for such important topics as the scaling of transport parameters with machine size.

2.2. The generalized master equation

From the definition of the particle motion pdf, equation (3), one may derive the time evolution equation of the particle density $n(x, t)$, i.e. the particle density distribution function multiplied by the total number of particles in the system [9–11]. The mathematically rigourous derivation presented in the cited papers extends previous similar results [12–15] to the situation in which p is a (possibly non-linear) function of both space and time. This extension opens up the CTRW framework to a host of possibilities that are of relevance to the study of transport in fusion plasmas; namely, situations in which the diffusion coefficient depends on certain system parameters (the relation between p and the diffusion coefficient will be given below). An example of this is the well-known local temperature dependence of the diffusion coefficient in fusion plasmas.

Without entering into the details of this derivation here, the resulting equation, given here for an unbounded system, is known as the generalized master equation (GME):

$$\frac{\partial n(x,t)}{\partial t} = \int \mathrm{d}x' \, p(x - x'; x', t) \frac{n(x',t)}{\tau_{\mathrm{D}}(x')} - \frac{n(x,t)}{\tau_{\mathrm{D}}(x)}. \tag{4}$$

This integro-differential equation is the main product of the CTRW approach, under the assumptions mentioned above. It is also the basis of the rest of this work. In the appropriate limit of local transport and in the fluid limit, discussed below, it reduces to the usual diffusion equations (typically of the form $\partial n/\partial t = D\partial^2 n/\partial x^2$), while allowing the use of Lévy distributions and the modelling of inhomogeneous and non-linear systems. In spite of the apparent complexity of this integro-differential equation, its interpretation is easy. It states that the rate of change of particles at location x and time t is given by adding the new particles that arrive and substracting those that leave, per time unit. The arriving particles originate from another location x' by a process with a probability given by $p(x - x', x', t)$, while the time scale governing the process is given by $\tau_D(x')$.

2.3. The fluid limit

To clarify the meaning of the GME, it is useful to consider its fluid limit [10, 16, 17]. Note that the integral in equation (4) is a convolution. Therefore, we proceed to Fourier space to obtain

$$\frac{\partial n(x,t)}{\partial t} = -\frac{n(x,t)}{\tau_D(x)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dx' \frac{n(x',t)}{\tau_D(x')}\int_{-\infty}^{\infty} dk' p(k', x'; t) e^{-ik'(x-x')}, \tag{5}$$

where $p(k, x'; t)$ is the Fourier transform of $p(x, x'; t)$.

Fourier-transforming this equation with respect to k, and noting that

$$\int_{-\infty}^{\infty} dx \exp(-i(k'-k)x) = 2\pi\delta(k'-k), \tag{6}$$

we obtain

$$\frac{\partial n(k,t)}{\partial t} = -\frac{n(k,t)}{\tau_D(k)} + \int_{-\infty}^{\infty} dx' \frac{n(x',t)}{\tau_D(x')} p(k, x'; t) e^{ikx'}. \tag{7}$$

For simplicity, let us assume that the particle step distribution p is a Gaussian, so that its Fourier transform is [9]

$$p(k, x; t) = \exp(-\sigma(x,t)^2 k^2). \tag{8}$$

The distribution p can be approximated by a Taylor expansion in k-space, assuming small σk:

$$p(k, x; t) \simeq 1 - \sigma(x,t)^2 k^2. \tag{9}$$

This approximation will therefore be valid in the limit of small σ (local transport) and small k (fluid limit).

Inserting this into equation (7) and Fourier-inverting, we obtain:

$$\frac{\partial n(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}(D(x,t)n(x,t)). \tag{10}$$

where we have defined $D(x,t) = \sigma(x,t)^2/\tau_D(x)$ as is the common practice.

2.4. Diffusion in a homogeneous system

In a homogeneous system, D is constant by definition and does not depend on (x, t). Then equation (10) becomes

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}. \tag{11}$$

Using particle conservation $\partial n/\partial t = -\partial\Gamma/\partial x$, this leads to Fick's law [18], which states that the particle flux Γ is proportional to the density gradient:

$$\Gamma_F = -D\frac{\partial n}{\partial x}, \tag{12}$$

(where Γ_F is the—Fickian—particle flux). Fick's law is a phenomenological relation which has been proved useful in many (nearly) homogeneous systems. Its success at modelling diffusive transport in such systems has been so general that often it is even applied to inhomogeneous systems, without however providing any detailed justification. We note that (fusion) plasmas, which are usually very strongly heated and fuelled, are far from equilibrium and far from homogeneous. The question whether Fick's law may be applied to such systems is not unjustified [19–22], and we will address this point in the next section.

2.5. Diffusion in an inhomogeneous system

The diffusion equation for the case that D is a function of (x, t) is equation (10). We note that this implies the existence of a flux

$$\Gamma_{FP} = -\frac{\partial}{\partial x}(Dn). \tag{13}$$

Observe the difference with equation (12). Equation (13) has long been known to describe the diffusion of a population of tracers subject to spatial inhomogeneities and is known as the FP diffusion law [23, 24].

Curiously though, Fick's law is very often adhered to even in inhomogeneous systems. As a consequence, it is then observed that the experimental data do not fit the theoretical model (e.g. sometimes the observed fluxes go in the direction of the gradient, i.e. uphill). To remedy this problem, a convection or drift is added to the Fickian flux, i.e.

$$\Gamma = -D\frac{\partial n}{\partial x} + V \cdot n. \tag{14}$$

This equation is usually sufficient to fit the data. However, often the interpretation of this drift is difficult, since its experimental value turns out to be much larger than can be accounted for by external forces and system asymmetries.

However, when comparing equation (14) with the FP equation (13), it is obvious that

$$V = -\frac{\partial D}{\partial x}, \tag{15}$$

i.e. according to Fokker–Planck V is related to the *inhomogeneity* of D. Thus, in strongly inhomogeneous systems, such as fusion plasmas, a (possibly large) part of the observed convection ('pinch') may be due to the FP contribution. Of course, this does not preclude other physical mechanisms, corresponding to the mentioned external forces and asymmetries [25], to also contribute to the total convection (e.g. the Ware pinch in fusion plasmas [26]).

2.6. An experimental approach

We will now proceed to test the existence of the FP pinch experimentally. Consider a one-dimensional system with co-ordinate $x \in \langle -\infty, \infty \rangle$. The system is characterized by two system parameters τ_D = constant and $\sigma(x) = \sigma_L + \Theta(x)(\sigma_R - \sigma_L)$, where $\Theta(x)$ is the Heaviside function. The subscripts 'L' and 'R' refer to left and right, respectively. Since $D = \sigma^2/\tau_D$, the diffusion coefficient D has a discontinuous jump at $x = 0$ ($D = D_L$ for $x < 0$ and $D = D_R$ for $x > 0$). Initially, we set $n(x, 0) = n_0$. The boundary conditions are that $\partial n/\partial x = 0$ for $x \to \pm\infty$. We attempt to find solutions to the diffusive transport equations.

First, consider the standard diffusive equation:

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial n}{\partial x}\right). \tag{16}$$

This equation does not evolve away from the initial state $n(x, 0) = n_0$, since evidently $\partial n/\partial x = 0$ everywhere, so $\partial n/\partial t = 0$. This is consistent with the idea that Fick's law does not produce any flux in the absence of gradients in the transported quantity. We note that this situation is problematic. Namely, consider the point $x = 0$. Particles some distance d to the left of $x = 0$ have a certain probability to reach the right system half in a given time interval, as specified via D_L. Likewise, particles at *the same* distance d to the right of $x = 0$ have a probability to reach the left system half (given via D_R). This probability not being equal, there is a net flux right (or left) of particles originating at the same distance d left and

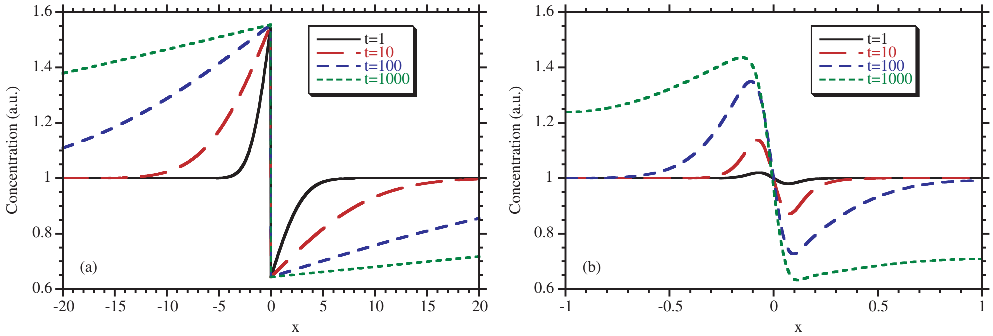

Figure 1. (*a*) Analytical solution to the fluid limit approximation of the GME, equations (18) and (19), with $n_0 = 1$, $D_L = 1.2$ and $D_R = 2.9$. (*b*) Numerical solution to the GME with the same parameters.

right of $x = 0$. This situation persists (with the same sign) for all distances d from $x = 0$, so a net flux through $x = 0$ must necessarily be generated. This intuitive result is at variance with the naive form of Fick's law.

Second, consider the FP diffusion law:

$$\frac{\partial n}{\partial t} = \frac{\partial^2}{\partial x^2}(Dn). \tag{17}$$

An analytical solution of this equation with the initial condition $n(x, 0) = n_0$ is

$$n_L(x,t) = n_0 - n_0 \frac{D_L - D_R}{\sqrt{D_L}(\sqrt{D_L} + \sqrt{D_R})}\left(1 + \mathrm{erf}\left(\frac{x}{2\sqrt{D_L t}}\right)\right) \tag{18}$$

(for $x < 0$) and

$$n_R(x,t) = n_0 + n_0 \frac{D_L - D_R}{\sqrt{D_R}(\sqrt{D_L} + \sqrt{D_R})}\left(1 - \mathrm{erf}\left(\frac{x}{2\sqrt{D_R t}}\right)\right) \tag{19}$$

(for $x > 0$), where $\mathrm{erf}(x)$ is the error function. Continuity of the flux $\partial(Dn)/\partial x$ has been imposed at $x = \pm\epsilon$ for $\epsilon \to 0$. These solutions are shown graphically in figure 1(*a*) with $n_0 = 1$, $D_L = 1.2$ and $D_R = 2.9$, for several values of t.

Finally, we consider the numerical solution to the GME, equation (4). This equation was solved directly, applying reflecting boundary conditions at $x = -1$ and $x = 1$ to avoid particle loss effects at the boundary. The result is shown in figure 1(*b*) for a case with $n_0 = 1$ and $D_L/D_R = 1.2/2.9$. Both the times t and the distances x indicated in the figure are in arbitrary units. The initially flat concentration profile ($n(x, 0) = \text{constant}$) is seen to evolve a gradual transition.

We have actually carried out this experiment in practice, using liquids of different viscosity with tracer particles [22], showing that the GME provides a very accurate description of diffusive transport in this inhomogeneous system. The FP diffusion equation provides a reasonably good approximation to the experimental results in those places where the inhomogeneity is not too strong (i.e. everywhere except near the discontinuity in D at $x = 0$).

This simple experiment clearly points in the direction that the FP diffusion equation is preferable to Fick's law for the modelling of diffusion in inhomogeneous systems. Evidently, the relevance of this simple experiment to the complexity of a fusion plasma may be questioned. However, the recent experiments performed at JET [27], in which tritium was used as a tracer to study transport coefficients, provided independent and spatially resolved information on both

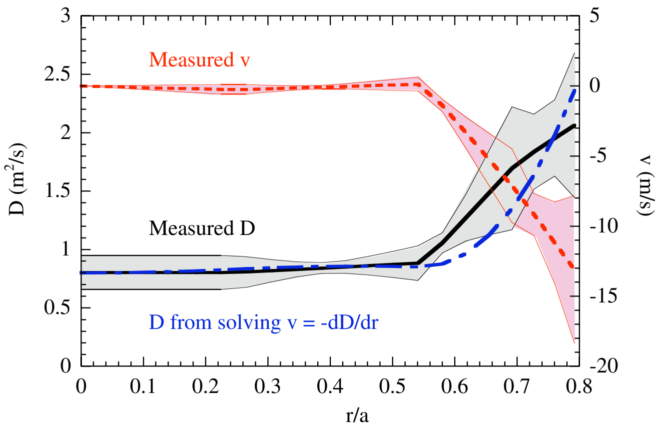

Figure 2. Radial profiles of D (——) and V (- - - -) as obtained from the trace tritium experiments at JET [27], with errors (shaded areas), and an estimate of D calculated from the FP prediction $V = -\partial D/\partial x$ (— · —), in good agreement with the measured D.

D and V (using equation (14) as a transport model), and the resulting profiles are in reasonably good accord with the FP prediction $V = -\partial D/\partial x$, both in sign and value (as shown in figure 2).

3. More complexity

Up to this point, the models emanating from the basic GME can still be considered 'classical' diffusion equations, and the merits of the CTRW/GME approach are mainly those of clarifying the assumptions behind and the limitations of the modelling framework. However, as noted before, the GME allows for far more complexity, and this we will explore in the following, always with an eye on its possible useful application in (fusion) plasma physics.

3.1. Lévy distributions and fractional differential equations

Why would one bother with Lévy distributions at all? The main reason for considering Lévy distributions for transport modelling is to capture some remarkable *scaling phenomena* of transport in fusion plasmas. As noted before, the Lévy distributions form a general class of which the Gaussian distribution is only a particular member. More precisely, the Lévy distributions encompass all limit distributions of *sums of independent, identically distributed* random variables. Whereas the Gaussian distribution decays rapidly, i.e. exponentially, for large x (namely, $p(x) \sim \exp(-x^2/\sigma^2)$), and thus generates *local* transport (as expressed by the FP equation), the non-Gaussian Lévy distributions decay slowly, i.e. algebraically, for large x (namely, $p(x) \sim \sigma^\alpha x^{-(1+\alpha)}$ with $0 < \alpha < 2$), and thus generate *non-local* transport, in which some 'transport event' has a small but non-negligible probability of traversing a large part of the system, and thus affecting the system at a large distance from its origin. Such non-local transport phenomena may have been observed in fusion plasmas [28–34], although there is always some margin for alternative interpretations.

Another indication comes from the fact that the scaling with system size (L) of experimental transport is typically not diffusive, e.g. the confinement time τ does not scale as L^2. If transport were dominated by a Lévy distribution with index α, then $\tau \sim L^\alpha$. Transitions between one and the other scaling behaviour as a function of the system drive are also possible, as shown in the example below.

Finally, Lévy distributions have been detected in numerical turbulence simulations [35] and experimental indications for their existence have been obtained [36, 37]. In combination, these indications inspire us to leave open the possibility of using Lévy distributions in the modelling of transport, which is easily accommodated in the CTRW/GME framework.

We note that the use of non-Gaussian Lévy pdfs does not impede one from making a fluid limit approximation in the same sense as discussed before. With Lévy pdfs, however, this leads to *fractional* differential equations (FDEs) [17] (and references therein). The study of FDEs is quite advanced [38], but a discussion of this issue is outside the scope of the current paper.

3.2. Criticality

A fundamental element of transport in fusion plasmas is the concept of criticality. The basic idea is that turbulence is activated when the gradient of, for example, pressure or some other relevant quantity exceeds some threshold, thus enhancing transport significantly. Such mechanisms would lead to the 'clamping' of the gradients to their critical values, provided the drive is sufficiently large, thus producing 'stiff' profiles, in accordance with the observed 'profile consistency' [39]. There are many experimental indications that such mechanisms are indeed operative in (fusion) plasmas [40, 41].

We note that this idea is evidently related to the central ideas of the self-organized criticality (SOC) paradigm [42], which provides an interesting alternative viewpoint of the transport problem [43–48]. Here also, unusual (i.e. non-diffusive) global scalings of the transport parameters are obtained, which bear some similarity to the effect of having Lévy distributions. Thus, from the global scaling behaviour of transport alone it will be difficult to distinguish between a critical mechanism and the action of Lévy distributions, unless the system can be driven to a fully supercritical state (where the criticality is deactivated), or the Lévy distributions can somehow be deactivated (perhaps by going to a high collisionality régime).

3.3. A toy model

The above ingredients lead us into largely uncharted terrain. To get a grip on the consequences of the cited mechanisms for transport, in previous work we have designed a one-field, one-dimensional toy model [9, 11]. The details of the calculations can be found in the cited references; here, we will merely cite the main results.

First, the GME is defined for a finite system. Without loss of generality, we set $0 \leqslant x \leqslant 1$. Particles that jump from a position within this range to a position outside are considered lost. This implies that the system suffers continuous edge losses, which must be compensated by a source term S, which we choose independent of time since here we intend to study steady-state only. For simplicity, we use a constant waiting time τ_D. The time evolution equation is therefore

$$\frac{\partial n}{\partial t} = \frac{1}{\tau_D} \int_0^1 dx' \, p(x - x'; x', t) n(x', t) - \frac{n(x, t)}{\tau_D} + S(x). \tag{20}$$

To capture the typical critical gradient behaviour, we choose the step distribution p in such a way that transport is 'local' (Gaussian) when the density gradient is below a given threshold and 'non-local' (Lévy) when the gradient is above it. When the source drive is sufficiently strong and the profile would rise above the critical value, the increased (Lévy-driven) transport will bring the gradient back to the critical value, thus 'clamping' the profile. We define p as

$$p(x - x'; x', t) = \zeta(x', t) P_{\rm sym}(x - x', 1, \sigma_1) + (1 - \zeta(x', t)) P_{\rm sym}(x - x', 2, \sigma_1), \tag{21}$$

where $P_{\rm sym}(x, \alpha, \sigma)$ is a Gaussian distribution when $\alpha = 2$ and a Cauchy distribution when $\alpha = 1$. The function ζ defines the criticality condition:

$$\zeta(x', t) = \Theta\left(\left|\frac{\partial n}{\partial x}\right| - \left(\frac{\partial n}{\partial x}\right)_c\right), \qquad (22)$$

where $\Theta(x)$ is the Heaviside function. As discussed in more detail in the cited references, this simple model has a number of remarkable properties.

First, with homogeneous fuelling $S(x) = S_0$, the mentioned profile clamping is observed for a range of values of S_0. At low fuelling levels, the system is subcritical everywhere and behaves as a standard diffusive model, and the particle confinement time τ scales with system size L as $\tau \sim L^2$, as expected. As S_0 increases, the profile is clamped to the critical value, first only at the edge, and as S_0 increases further, an ever larger part of the profile becomes critical. As a consequence, the confinement time τ drops; this effect is remarkably similar to the 'power degradation effect' observed in many fusion plasma experiments.

Second, as S_0 increases, the scaling of τ with system size makes a simultaneous transition from $\tau \sim L^2$ to $\tau \sim L$, which is reminiscent of the transition Bohm $\rightarrow$ gyro-Bohm observed in fusion plasmas [49].

Third, still with homogeneous fuelling, the stiffness of the profile produces an extremely rapid propagation of perturbations, some three orders of magnitude faster than one might expect from the diffusive time-scale, very similar to what is observed with pulse propagation studies in fusion experiments [28, 50–52].

Fourth, with an off-axis fuelling profile $S(x) = S_0 f(x)$, where $f(x)$ is a positive function with unit integral and with peaks at $x = 0.3$ and $x = 0.7$ [11], the density profile is initially flat in the core region (at low values of S_0) but peaks as the fuelling level S_0 is increased. Again, this is strikingly similar to what is observed in some actual experiments (e.g. [53]), and very unexpected from local diffusive models. As discussed in our previous work, the explanation of this peaking is essentially linked to the criticality of the transport model.

An interesting observation, in view of recent discussions regarding density profile peaking, is that the profile peaking in the toy model depends on the relative importance of the sub-critical (diffusive) transport channel in overall transport. In other words, the peaking is reduced and the profile becomes less stiff when the diffusive transport channel is enhanced. This is in accord with recent observations in fusion plasmas [54, 55] that the peaking decays as the collisionality parameter $\nu_{\rm eff}$ is increased.

In conclusion, this very simple toy model, based on the CTRW/GME framework and incorporating transport mechanisms that are inspired, on the one hand, by physics that is commonly suspected to be operative (critical gradients) [56], and on the other, by processes whose relevance is not yet proved (Lévy distributions), produces very interesting behaviour that bears a striking similarity with the behaviour of actual fusion plasmas.

3.4. Towards a full transport model

The toy model presented in the preceding section is limited in various aspects. The first limitation that needs to be overcome is the one-field aspect of the model (only the density, n, is modelled). Plasmas are characterized, to first approximation, by at least two fields; namely, density (n) and temperature (T).

To extend the model in this sense, it is useful to recall that the GME (equation (4)) expresses the principle of *conservation of density*. Thus, it is natural to assume that a similar GME might need to be introduced for any other conserved quantity, such as the plasma energy $E = \frac{3}{2}nT$.

This immediately leads us to postulate the following equation:

$$\frac{\partial E(x,t)}{\partial t} = \int \mathrm{d}x' p_E(x-x';x',t) \frac{E(x',t)}{\tau_{\mathrm{D}}(x')} - \frac{E(x,t)}{\tau_{\mathrm{D}}(x)}. \tag{23}$$

Indeed we now have a set of two equations describing the evolution of two fields (namely, n and E, or, equivalently, n and T). Note the appearance of a second step probability distribution, p_E. All the physics of energy transport is expressed via this distribution, just like the physics of particle transport was expressed via p.

The crux of the matter is now to choose p and p_E in such a way that reasonable models for particle and energy transport are obtained. Such choices must be motivated by the (kinetic) theory of transport and the theory of plasma instabilities. Thus, it may be expected that both ∇p and $\eta = \nabla(\ln T)/\nabla(\ln n)$ play a role in the definition of critical gradients. Work in this direction is currently underway.

4. Conclusions

In this work, we have reviewed certain aspects of the CTRW modelling framework and suggest that the corresponding GME might provide a useful tool for the numerical modelling of diffusion of particles and energy in magnetically confined plasmas.

In the appropriate limit, the GME corresponds to the well-known FP diffusion equation in inhomogeneous systems, which reduces to Fick's law under the additional assumption that the system is homogeneous. Since fusion plasmas are strongly inhomogeneous, this suggests that the FP diffusion equation may be more appropriate than Fick's law for modelling transport. Part of the observed inward convection ('pinch') may then simply be ascribed to the inhomogeneity of the system.

The ample scope of the GME also permits a mathematically sound approach to more complex transport issues, such as the incorporation of critical gradients and non-local transport mechanisms (using Lévy distributions). A toy model incorporating these ingredients was shown to possess behaviour that bears a striking similarity to certain unusual phenomena observed in fusion plasmas. It is hoped that a suitable extension of the current GME framework will eventually lead to a full-fledged transport model for fusion plasmas. As mentioned before, its construction should be guided partly by the kinetic theory of particle motion and partly by the theory of plasma instabilities. These theories should delimit, in a statistical sense, the probability distributions needed to represent the underlying microscopic dynamics correctly. Further adjustment of the obtained model should then be possible by direct comparison with experimental results.

Acknowledgments

The authors would like to thank the JET EFDA contributors and, in particular, A D Whiteford, for kindly providing access to the data shown in figure 2. Interesting discussions and a fruitful collaboration with P D Bons are acknowledged. This research was sponsored in part by DGICYT (Dirección General de Investigaciones Científicas y Tecnológicas) of Spain under Project No FTN2003-08337-C04-02. Part of this research has been carried out at Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the US Department of Energy under contract number DE-AC05-00OR22725. Part of this research was supported by Spanish DGES Projects Nos FTN2003-04587 and FTN2003-08337-C04-01.

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